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Kohn's Theorem, Larmor's Equivalence Principle and the Newton-Hooke Group

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Abstract

We consider non-relativistic electrons, each of the same charge to mass ratio, moving in an external magnetic field with an interaction potential depending only on the mutual separations, possibly confined by a harmonic trapping potential. We show that the system admits a "relativity group" which is a one-parameter family of deformations of the standard Galilei group to the Newton-Hooke group which is a Wigner-İnönü contraction of the de Sitter group. This allows a group-theoretic interpretation of Kohn's theorem and related results. Larmor's Theorem is used to show that the one-parameter family of deformations are all isomorphic. We study the "Eisenhart" or "lightlike" lift of the system, exhibiting it as a pp-wave. In the planar case, the Eisenhart lift is the Brdička-Eardley-Nappi-Witten pp-wave solution of Einstein-Maxwell theory, which may also be regarded as a bi-invariant metric on the Cangemi-Jackiw group.

1 Introduction

Since at least the time of Galieo Galilei, physicists have realised the importance of understanding how physical quantities change under changes of the frame of reference. With the development of Special Relativity, attention became focussed on the underlying group, and its invariants. Einstein's construction of the theory of General Relativity was based on the related idea that the paths of "freely falling particles" should be universal, that is to say, independent of their mass or other properties. However, as we aim to show in this paper, the utility of this viewpoint is not restricted to high energy physics or general relativity, but may also be applied with advantage to situations where the energies and speeds of the particles are moderate compared with that of light, and the gravitational fields are weak. All that matters is that the dynamical behaviour of the particles should be universal and that some sort of "Equivalence Principle" hold.

These requirements are met in an extremely useful and well studied model in condensed matter physics, consisting of N electrons, each of mass m and charge e, moving in a uniform background magnetic field $\mathbf{B}$. The electrons are assumed to interact with each other via a potential V that depends only on their $\frac{1}{2}N(N-1)$ relative positions: $V = \sum_{a\neq b} V(\mathbf{r}_a - \mathbf{r}_b)$, $a, b = 1, 2, , N$. A typical example would be a Coulomb potential, which is actually central, i.e. which depends only on the relative distances $r_{ab} = |\mathbf{r}_a - \mathbf{r}_b|$. However, for what follows V need not be central. In addition, especially in the case of "quantum dots" [1], the electrons may be confined by an additional harmonic or "parabolic" trapping potential of the form $\frac{1}{2}m\omega^2 \sum_a \mathbf{r}_a^2$, or more generally, an anisotropic oscillator potential. The classical Lagrangian in the isotropic case is therefore

$$L = \sum_a \Big(\frac{1}{2} m\dot{\mathbf{r}}_a^2 + e\mathbf{A}_a \cdot \dot{\mathbf{r}}_a - \frac{1}{2} m\omega^2\, \mathbf{r}_a^2 \quad \Big) - \sum_{a\neq b} V(\mathbf{r}_a - \mathbf{r}_b)\,, \tag{1.1}$$

and the classical equations of motion are

$$m\ddot{\mathbf{r}}_a = e\dot{\mathbf{r}}_a \times \mathbf{B} - m\omega^2 \mathbf{r}_a - \boldsymbol{\nabla}_a V\,, \tag{1.2}$$

where $\boldsymbol{\nabla}_a = \frac{\partial}{\partial \mathbf{r}_a}$, $\mathbf{A}_a = \mathbf{A}(\mathbf{r}_a)$ and $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. Since $\mathbf{B}$ is assumed uniform, $\mathbf{A}$ may be taken to be linear in $\mathbf{r}$, and in what follows we shall adopt the gauge for which

$$\mathbf{A}(\mathbf{r}) = -\tfrac{1}{2}\mathbf{r} \times \mathbf{B}\,. \tag{1.3}$$

The system of equations (1.2) has an extremely important property, first pointed out by Kohn [2] in the case without trapping potential (i.e. with $\omega = 0$), and referred to as *Kohn's*

Theorem. This theorem states that one may split off the "centre of mass motion" in such a way that the interaction potential $V(\mathbf{r}_a - \mathbf{r}_b)$ does not enter. In other words, the centre of mass degree of freedom, $\mathbf{r} = \frac{1}{N}\sum_a \mathbf{r}_a$, completely decouples from the $(N-1)$ independent relative degrees of freedom in a universal way. For this property to continue to hold when a trapping potential is present, the potential must be the sum of identical quadratic terms for each electron, although each quadratic term need not be, as it is in (1.1), isotropic in $\mathbf{r}$.

One way to obtain Kohn's result is to substitute the identity

$$\sum_a \dot{\mathbf{r}}_a^2 = \frac{1}{N}\Big(\sum_a \dot{\mathbf{r}}_a\Big)^2 + \frac{1}{N}\sum_{a<b}(\dot{\mathbf{r}}_a - \dot{\mathbf{r}}_b)^2 \tag{1.4}$$

into the Lagrangian (1.1). Alternatively, one may note that the equations of motion (1.2) admit a symmetry

$$\mathbf{r}_a \to \mathbf{r}_a + \mathbf{a}(t) \tag{1.5}$$

where

$$\ddot{\mathbf{a}} = \frac{e}{m}\dot{\mathbf{a}} \times \mathbf{B} - \omega^2 \mathbf{a} \tag{1.6}$$

For this reason, Kohn's theorem is often ascribed to the Galilei invariance of the system. However this is not actually correct since the solutions of (1.6) are not of the form $\mathbf{a} = \mathbf{b} + \mathbf{u}t$. Moreover, the presence of a non-vanishing magnetic field $\mathbf{B}$ breaks the $SO(3)$ rotational symmetry down to the $SO(2)$ subgroup of rotations about the direction of the magnetic field. Nevertheless, for fixed ω and $B = |\mathbf{B}|$, the second-order differential equation (1.6) does have a 6-parameter set of solutions which in general defines a 6-parameter symmetry group. This six-dimensional group of generalised translations and boosts is abelian, just as in the Galilei case. However since the transformations, i.e. the solutions of (1.6), depend non-trivially on time, when time translations are taken into account we get a seven-dimensional non-abelian group. Thus the relevant symmetry group is not the translation, boost and time-translation subgroup of the Galilei group, but appears to belong instead to a continuous one-parameter family of deformations depending on the dimensionless ratio $eB/(m\omega)$. If the residual rotations are added we get an eight-dimensional group. However the apparent dependence on the dimensionless parameter $eB/(m\omega)$ is illusory, since by an application of a well known result of Larmor [3], the magnetic field B may be eliminated by passing to a rotating frame with angular velocity $\boldsymbol{\varpi}$ given by

$$\varpi = -\frac{eB}{2m}, \tag{1.7}$$

at the expense of introducing a quadratic potential.

To see this in detail, note that if we pass to a frame that is rotating with constant angular velocity $\boldsymbol{\varpi}$ then for any vector $\mathbf{x}$,

$$\dot{\mathbf{x}} = \mathbf{x}' + \boldsymbol{\varpi} \times \mathbf{x} \,, \tag{1.8}$$

where a $\dot{}$ indicates a time derivative of the components of $\mathbf{x}$ in an inertial or space-fixed frame, and a $'$ denotes a time derivative of the components in the rotating frame.

The equations of motion become

$$m\mathbf{r}''_a = e\mathbf{r}'_a \times \Big(\mathbf{B} + \frac{2m}{e}\boldsymbol{\varpi}\Big) - m\omega^2\mathbf{r}_a + e(\boldsymbol{\varpi}\times\mathbf{r}_a)\times\mathbf{B} - m\boldsymbol{\varpi}\times(\boldsymbol{\varpi}\times\mathbf{r}_a) - \boldsymbol{\nabla}_a V(\mathbf{r}_a - \mathbf{r}_b)\,. \tag{1.9}$$

The equations of motion (1.2) are unchanged in form if we make the replacement

$$\mathbf{B} \rightarrow \mathbf{B} + \frac{2e}{m}\boldsymbol{\varpi} \tag{1.10}$$

and modify the trapping force, a result usually called Larmor's theorem. However

$$(\boldsymbol{\varpi} \times \mathbf{r}_a) \times \mathbf{B} = \frac{1}{2}\boldsymbol{\nabla}_a(\mathbf{r}_a \cdot \mathbf{r}_a)(\boldsymbol{\varpi}\cdot\mathbf{B}) - (\mathbf{r}_a \cdot \mathbf{B})\boldsymbol{\varpi} \tag{1.11}$$

and

$$\boldsymbol{\nabla}_a \ \times \big((\mathbf{r}_a \cdot \mathbf{B})\,\boldsymbol{\varpi}\big) = \mathbf{B} \times \boldsymbol{\varpi}\,, \tag{1.12}$$

and so the motion in the modified trapping force is not conservative unless either $\boldsymbol{\varpi}\times\mathbf{B} = 0$ or the motion is perpendicular to the magnetic field, $\mathbf{r}_a \cdot \mathbf{B} = 0$. In the latter case, the passage to the rotating frame is a symmetry relating equivalent systems connected by

$$B \rightarrow \tilde{B} = B + \frac{2m}{e}\varpi \qquad \omega^2 \rightarrow \tilde{\omega}^2 = \omega^2 - \varpi^2 - \frac{e}{m}B\varpi = \omega^2 - \big(\varpi + \frac{eB}{2m}\big)^2 + \frac{e^2B^2}{4m^2}\,. \tag{1.13}$$

Note that under Larmor's transformation

$$\Omega^2 = \omega^2 + \frac{e^2B^2}{4m^2} = \tilde{\Omega}^2 = \tilde{\omega}^2 + \frac{e^2\tilde{B}^2}{4m^2}\,, \tag{1.14}$$

and so Ω^2 is unchanged.

In the planar case, this is just a modification of the existing quadratic trapping potential, and therefore one may always eliminate the magnetic field. One may also check that if one wishes, rather than eliminating the magnetic field, one may use a Larmor transformation to eliminate the trapping potential.

Taking the first option, the symmetry group of the system is now easy to identify. It is the *Newton-Hooke group* in $(2+1)$ dimensions. In $(3+1)$ dimensions the Newton-Hooke group is one of the possible *kinematic groups* first classified by Levy-Leblond and

Bacry [4, 5], which may be regarded as a small-velocity limit, or Wigner-İnönü contraction [6], of the of the anti-de Sitter group $SO(3,2)$. In effect, the trapping potential behaves like a universal cosmic attraction associated with a negative cosmological constant $\Lambda = -3\omega^2/c^2$ [12]. The Newton-Hooke group has been applied to non-relativistic cosmology, both classically and at the quantum level[13]. Thus one is tempted to wonder whether the deformations where the magnetic field is included may also have a cosmological application to a Goedel-type rotating universe with a cosmological term.

In the planar case the relevant Newton Hooke group is six-dimensional, and it is a Wigner-İnönü contraction of $SO(2,2)$.

One purpose of the present paper is to explore the properties of the deformed group in more detail, and to use it to obtain the quantum mechanical spectrum of the centre of mass motion for general $\mathbf{B}$ and ω directly, by using group theory. This was partially done by Kohn in his original paper [2] in the case $\omega = 0$, by using an operator method. The full spectrum of a particle in a magnetic field with harmonic potential was obtained by Fock [7] and by Darwin [8] long ago, by solving explicitly for the wave functions and using the properties of the associated Laguerre polynomials. We shall obtain the spectrum without solving for the radial functions, by using oscillator methods.

It is well known that at the quantum mechanical level, the classical 10-dimensional Galilei group enlarges to Bargmann's 11-dimensional central extension [9]. e find that a similar phenomenon occurs in our case. (For an alternative discussion of central extensions of the $(2+1)$-dimensional Newton-Hooke group, from the point of view of nonlinear realisations, see [10].) In order to obtain the quantum mechanical symmetry group we first turn to a classical Hamiltonian treatment, obtaining the canonical generators or "moment maps" of the seven-dimensional symmetry group. It is well known that the Lie algebra of a transformation group acting on phase space will not in general coincide with the Poisson algebra of its moment maps, because of the possibility of central terms. That is precisely what we find in our case. Upon quantisation of the moment maps by replacing the canonical momentum $\boldsymbol{\pi}$ by $-i\hbar\boldsymbol{\nabla}$ acting on the quantum mechanical wave function, we indeed find that it is the centrally-extended Poisson algebra that applies at the quantum level.

In the planar case, we obtain in this way a seven-dimensional extended group which is isomorphic to $\big(\mathcal{CJ}\otimes\mathcal{CJ}\big)/N$, where $\mathcal{CJ}$ is Cangemi and Jackiw's central extension of the two-dimensional Euclidean group [11], N being the central extension.

A nice geometric way of understanding the central extensions of non-relativistic symmetry groups acting on a $(d+1)$-dimensional non-relativistic spacetime is via a "light-

like reduction" of a Lorentzian metric on a $(d+2)$-dimensional spacetime that admits a covariantly-constant null Killing vector field [14, 15]. According to a result of Eisenhart [16], a rather general class of mechanical systems may be lifted so that their motion is a "shadow" of a null geodesic in $d+2$ dimensions. In our case $d = 3N$, and we carry out the lift and describe some of the properties of the higher-dimensional spacetime. The Eisenhart lift of the centre of mass motion in the planar case turns out to be a Nappi-Witten pp-wave [17], which has arisen in string theory [18, 19, 20, 21, 22]. This particular four-dimensional pp-wave is homogeneous, with the seven-dimensional Newton-Hooke group as its symmetry. In fact it may be identified as the group manifold of the universal cover of the Cangemi-Jackiw group $\mathcal{CJ}$.

The organisation of the paper is as follows. In section 2 we shall describe the action of the classical relativity group of our model on its non-relativistic spacetime. We construct its infinitesimal generators as vector fields on the spacetime and evaluate their Lie brackets and hence determine the associated Lie algebra. In section 3 we provide a brief account of the modifications brought about by a uniform electric field. In section 4 we pass to a classical Hamiltonian treatment and find that the Poisson algebra of functions on phase space differs from the classical Lie algebra by central terms. In section 5 we pass to the quantum theory and find that the central terms are present in the quantum algebra. We use the algebra to obtain the wave functions and energy eigenvalues. In section 6 we lift our system from configuration space to a Lorentzian spacetime with two extra dimensions admitting a null Killing vector, for which the classical motion lifts to that of a null geodesic. In section 7 we identify the light-like lift in the planar case to be the Brdička-Eardley-Nappi-Witten pp-wave spacetime.

2 The Action on Non-Relativistic Spacetime

In this section we give the solutions of (1.6) and use them to obtain the action of the seven-dimensional group on the $(3+1)$-dimensional non-relativistic spacetime $\mathcal{M}_{\mathrm{rel}}$ whose coordinates are $(t, \mathbf{r})$. This allows us to obtain their infinitesimal generators as vector fields acting on $\mathcal{M}_{\mathrm{rel}}$. The Lie algebra is then obtained by taking Lie brackets.

Consider the system (1.6) with a uniform magnetic field. Taking $\mathbf{B}$ to lie in the z direction, and defining

$$\nu = \frac{eB}{2m}\,, \qquad \Omega = \sqrt{\omega^2 + \nu^2}\,, \tag{2.1}$$

where ν is the Larmor frequency, the general solution of (1.6) is given by

$$\begin{aligned}
a_x &= \alpha_1 \cos(\Omega+\nu)t + \alpha_2 \sin(\Omega+\nu)t + \beta_1 \cos(\Omega-\nu)t - \beta_2 \sin(\Omega-\nu)t\,, \\
a_y &= \alpha_2 \cos(\Omega+\nu)t - \alpha_1 \sin(\Omega+\nu)t + \beta_2 \cos(\Omega-\nu)t + \beta_1 \sin(\Omega-\nu)t\,, \\
a_z &= \alpha_3 \cos\omega t + \beta_3 \sin\omega t\,,
\end{aligned} \tag{2.2}$$

where α_i and β_i are 6 arbitrary constants. Thus we may read off the 6 associated generators:

$$\begin{aligned}
\mathrm{K}_1 &= \cos(\Omega+\nu)t\,\frac{\partial}{\partial x} - \sin(\Omega+\nu)t\,\frac{\partial}{\partial y}\,, \\
\mathrm{K}_2 &= \sin(\Omega+\nu)t\,\frac{\partial}{\partial x} + \cos(\Omega+\nu)t\,\frac{\partial}{\partial y}\,, \\
\mathrm{K}_3 &= \cos(\Omega-\nu)t\,\frac{\partial}{\partial x} + \sin(\Omega-\nu)t\,\frac{\partial}{\partial y}\,, \\
\mathrm{K}_4 &= -\sin(\Omega-\nu)t\,\frac{\partial}{\partial x} + \cos(\Omega-\nu)t\,\frac{\partial}{\partial y}\,, \\
\mathrm{K}_5 &= \cos\omega t\,\frac{\partial}{\partial z}\,, \\
\mathrm{K}_6 &= \frac{\sin\omega t}{\omega}\,\frac{\partial}{\partial z}\,.
\end{aligned} \tag{2.3}$$

These all commute. If we define also $\mathrm{H} = \partial/\partial t$, then we have the commutators

$$\begin{aligned}
[\mathrm{H},\mathrm{K}_1] &= -(\Omega+\nu)\,\mathrm{K}_2\,, \qquad [\mathrm{H},\mathrm{K}_2] = (\Omega+\nu)\,K_1\,, \\
[\mathrm{H},\mathrm{K}_3] &= (\Omega-\nu)\,\mathrm{K}_4\,, \qquad [\mathrm{H},\mathrm{K}_4] = -(\Omega-\nu)\,\mathrm{K}_3\,, \\
[\mathrm{H},\mathrm{K}_5] &= -\omega^2\,\mathrm{K}_6\,, \qquad [\mathrm{H},\mathrm{K}_6] = \mathrm{K}_5\,.
\end{aligned} \tag{2.4}$$

In the limit when $\omega = 0$, the six $\mathrm{K}_{(\alpha)}$ generators become

$$\begin{aligned}
\mathrm{K}_1 &= \cos 2\nu t\,\frac{\partial}{\partial x} - \sin 2\nu t\,\frac{\partial}{\partial y}\,, \\
\mathrm{K}_2 &= \sin 2\nu t\,\frac{\partial}{\partial x} + \cos 2\nu t\,\frac{\partial}{\partial y}\,, \\
\mathrm{K}_3 &= \frac{\partial}{\partial x}\,, \qquad K_4 = \frac{\partial}{\partial y}\,, \\
\mathrm{K}_5 &= \frac{\partial}{\partial z}\,, \qquad \mathrm{K}_6 = t\,\frac{\partial}{\partial z}\,.
\end{aligned} \tag{2.5}$$

Note that even if $\omega^2 = 0$, and we do not obtain the two-dimensional Galilei group,

The equations of motion are invariant under rotation about the direction of the magnetic field $\mathbf{B}$. This is generated by

$$\mathrm{J} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\,. \tag{2.6}$$

Clearly J commutes with H, K_5 and K_6. It acts by rotations on (K_1, K_2) and (K_3, K_4):

$$\begin{aligned} \left[J, K_1\right] &= -K_2 \qquad \left[J, K_2\right] = K_1 \\ \left[J, K_3\right] &= -K_4 \qquad \left[J, K_4\right] = K_3 \,. \end{aligned} \tag{2.7}$$

The six-dimensional sub-algebra spanned by $(H, J, K_1, K_2, K_3, K_4)$ appears to depend on the two constants ν and Ω, but as mentioned earlier, by Larmor's theorem, the apparent dependence on ν is illusory. To see this explicitly we may pass to coordinates $\tilde{x}, \tilde{y}$ in a rotating frame by setting

$$x + iy = e^{i\varpi t}(\tilde{x} + i\tilde{y}) \tag{2.8}$$

so that

$$a_1 + ia_2 = e^{i\varpi t}(\tilde{a}_1 + i\tilde{a}_2) \tag{2.9}$$

and find that in the new coordinates Ω is unchanged but ν is replaced by $\nu + \varpi$. By an appropriate choice of the angular velocity ϖ of our rotating frame we may give the Larmor frequency ν any value we wish. Despite this, we we shall retain both parameters in the formulae that follow.

3 Electric Fields

We shall now briefly review the situation when a uniform time-independent electric field $\mathbf{E}$ is present, since this introduces new some features. The equation of motion is now

$$\ddot{\mathbf{r}}_a = \frac{e}{m}\mathbf{E} + \frac{e}{m}\dot{\mathbf{r}}_a \times \mathbf{B} - \omega^2 \mathbf{r}_a - \frac{1}{m}\boldsymbol{\nabla}_a V \tag{3.1}$$

The point is that while (3.1) is is still invariant under the deformed Galilei group, i.e. under

$$\mathbf{r}_a \to \mathbf{r}_a + \mathbf{a}(t)\,, \tag{3.2}$$

one may enhance the symmetry group of the equation if one allows a transformation of the electric field $\mathbf{E}$. Consider, to begin with, the case with an electric field but no magnetic field and no trapping potential, so that

$$\ddot{\mathbf{r}}_a = \frac{e}{m}\mathbf{E} - \frac{1}{m}\boldsymbol{\nabla}_a V\,. \tag{3.3}$$

Because the ratios of charge to mass of all the particles are identical, the equation of motion is now invariant if one transforms $\mathbf{r}_a$ as

$$\mathbf{r}_a(t) \to \mathbf{r}_a(t) + \frac{1}{2}\mathbf{C}t^2 \tag{3.4}$$

and one transforms the electric field as

$$\mathbf{E} \to \mathbf{E} - \frac{m}{e}\mathbf{C}\,. \tag{3.5}$$

Indeed if one picks $\mathbf{C} = \frac{e}{m}\mathbf{E}$, one may eliminate the electric field altogether. Clearly this is the analogue of Einstein's equivalence principle. The additional vector fields are

$$\mathrm{C}_i = \frac{1}{2}t^2\nabla_i \tag{3.6}$$

and

$$[H, C_i] = t\nabla_i \tag{3.7}$$

where the right-hand side of (3.7) is a Galilean boost.

The inclusion of a magnetic field changes things. The equation of motion is now

$$\ddot{\mathbf{r}}_a = \frac{e}{m}\mathbf{E} + \frac{e}{m}\dot{\mathbf{r}}_a \times \mathbf{B} - \frac{1}{m}\boldsymbol{\nabla}_a V\,, \tag{3.8}$$

and a boost

$$\mathbf{r}_a(t) \to \mathbf{r}_a(t) + \mathbf{u}t \tag{3.9}$$

induces a "non-relativistic Lorentz transformation"

$$\mathbf{B} \to \mathbf{B}\,, \qquad \mathbf{E} \to \mathbf{E} - \mathbf{u} \times \mathbf{E}\,. \tag{3.10}$$

Indeed if we choose the drift velocity $\mathbf{v}_d$ to satisfy

$$\mathbf{E} + \mathbf{v}_d \times \mathbf{B} = 0\,, \tag{3.11}$$

we can eliminate the electric field altogether. This is essentially the classical Hall effect.

The presence of the trapping potential further complicates matters since a translation

$$\mathbf{r}_a(t) \to \mathbf{r}_a(t) + \mathbf{b} \tag{3.12}$$

induces the transformation

$$\mathbf{E} \to \mathbf{E} - \frac{m\omega^2}{e}\mathbf{b} \tag{3.13}$$

and if we choose

$$\mathbf{b} = \frac{e}{m\omega^2}\mathbf{E}\,, \tag{3.14}$$

we can also eliminate the electric field altogether.

The upshot of this discussion would seem to be that by a suitable symmetry transformation we can always eliminate any uniform electric field and we have seen above that we may remove any uniform magnetic field. In what follows we shall not consider electric fields further.

4 Classical Hamiltonian Treatment

Starting from the Lagrangian (1.1), and the canonical momentum

$$\boldsymbol{\pi}_a = m\dot{\mathbf{r}}_a + e\mathbf{A}_a = \mathbf{p}_a + e\mathbf{A}_a\,, \tag{4.1}$$

the Poisson brackets are

$$\{x_{ia}, \pi_{jb}\} = \delta_{ij}\delta_{ab}\,, \qquad \{x_{ia}, x_{jb}\} = 0\,, \qquad \{\pi_{ia}, \pi_{jb}\} = 0\,, \tag{4.2}$$

and

$$\{x_{ia}, p_{jb}\} = \delta_{ij}\delta_{ab}\,, \qquad \{p_{ia}, p_{ib}\} = e\delta_{ab}\epsilon_{ijk}B_k\,. \tag{4.3}$$

The Hamiltonian is given by

$$\begin{aligned} H &= \frac{1}{2m}\sum_a (\boldsymbol{\pi}_a - e\mathbf{A}_a)^2 + \tfrac{1}{2}m\omega^2 \sum_a \mathbf{r}_a^2 + \sum_{a<b} V(\mathbf{r}_a - \mathbf{r}_b)\,,\\ &= \frac{1}{2m}\sum_a \mathbf{p}_a^2 + \tfrac{1}{2}m\omega^2 \sum_a \mathbf{r}_a^2 + \sum_{a<b} V(\mathbf{r}_a - \mathbf{r}_b)\,,\\ &= \mathcal{H} + \mathcal{H}_{\rm rel}\,, \end{aligned} \tag{4.4}$$

where the centre-of-mass and relative Hamiltonians are given by

$$\mathcal{H} = \frac{1}{2M}\,\mathbf{p}^2 + \tfrac{1}{2}M\omega^2\,\mathbf{r}^2\,, \tag{4.5}$$

$$\mathcal{H}_{\rm rel} = \sum_{a,b}\Big(\frac{1}{2mN}(\mathbf{p}_a - \mathbf{p}_b)^2 + \frac{1}{2N}\,m\omega^2(\mathbf{r}_a - \mathbf{r}_b)^2 + V(\mathbf{r}_a - \mathbf{r}_b)\Big)\,, \tag{4.6}$$

N is the number of electrons, and $M = mN$ is their total mass. Note that we have defined the centre-of-mass coordinates $\mathbf{r}$, and the centre-of-mass momentum $\mathbf{p}$ by

$$\mathbf{r} = \frac{1}{N}\sum_a \mathbf{r}_a\,, \qquad \mathbf{p} = \sum_a \mathbf{p}_a\,. \tag{4.7}$$

Similarly, we define the centre-of-mass canonical momentum by

$$\boldsymbol{\pi} = \sum_a \boldsymbol{\pi}_a \tag{4.8}$$

and so we shall have the Poisson bracket relations

$$\{x_i, \pi_j\} = \delta_{ij}\,, \qquad \{x_i, x_j\} = 0\,, \qquad \{\pi_i, \pi_j\} = 0\,. \tag{4.9}$$

We also have

$$\{p_i, p_j\} = eN\epsilon_{ijk}B_k\,, \qquad \{x_i, p_j\} = \delta_{ij}\,, \tag{4.10}$$

and

$$\dot{\mathbf{p}} = \frac{e}{m}\mathbf{p} \times \mathbf{B}\,. \tag{4.11}$$

Note that eN is the total charge.

In the "symmetric" gauge (1.3) that we are using, we have

$$p_x = \pi_x + \nu M y\,, \qquad p_y = \pi_y - \nu M x\,, \qquad p_z = \pi_z\,. \tag{4.12}$$

The generators $\mathrm{K}_{(\alpha)}$ defined in (2.3) act on the x^i and p_i, according to

$$\delta_{(\alpha)} x^i = \mathrm{K}^i_{(\alpha)}\,, \qquad \delta_{(\alpha)} p_i = m\delta\dot{x}^i = M\dot{\mathrm{K}}^i_{(\alpha)}\,, \tag{4.13}$$

where the dot means a derivative with respect to t. We may therefore read off the corresponding moment maps $\kappa_{(\alpha)}$, i.e. the functions on phase space such that

$$\delta_{(\alpha)}\, x^i = \{x^i, \kappa_{(\alpha)}\}\,, \qquad \delta_{(\alpha)}\, p_i = \{p_i, \kappa_{(\alpha)}\}\,. \tag{4.14}$$

We find that they are given by

$$\begin{aligned}
\kappa_1 &= (\pi_x + M\Omega y)\cos(\Omega+\nu)t - (\pi_y - M\Omega x)\sin(\Omega+\nu)t\,,\\
\kappa_2 &= (\pi_x + M\Omega y)\sin(\Omega+\nu)t + (\pi_y - M\Omega x)\cos(\Omega+\nu)t\,,\\
\kappa_3 &= (\pi_x - M\Omega y)\cos(\Omega-\nu)t + (\pi_y + M\Omega x)\sin(\Omega-\nu)t\,,\\
\kappa_4 &= -(\pi_x - M\Omega y)\sin(\Omega-\nu)t + (\pi_y + M\Omega x)\cos(\Omega-\nu)t\,,\\
\kappa_5 &= \pi_z\,\cos\omega t + M\omega z\,\sin\omega t\,,\\
\kappa_6 &= \pi_z\,\frac{\sin\omega t}{\omega} - Mz\,\cos\omega t\,.
\end{aligned} \tag{4.15}$$

It is straightforward to see that the moment maps have the following non-vanishing Poisson brackets:

$$\{\kappa_1, \kappa_2\} = 2M\Omega\,, \qquad \{\kappa_3, \kappa_4\} = -2M\Omega\,, \qquad \{\kappa_5, \kappa_6\} = M\,. \tag{4.16}$$

The centre-of-mass Hamiltonian (4.5) is given by

$$\mathcal{H} = \frac{1}{2M}\Big[(\pi_x + M\nu y)^2 + (\pi_y - M\nu x)^2 + \pi_z^2\Big] + \tfrac{1}{2}M\omega^2(x^2+y^2+z^2)\,. \tag{4.17}$$

It can be verified that the Poisson brackets of the Hamiltonian with the moment maps are given by

$$\begin{aligned}
\{\mathcal{H}, \kappa_1\} &= -(\Omega+\nu)\,\kappa_2\,, \qquad \{\mathcal{H}, \kappa_2\} = (\Omega+\nu)\,\kappa_1\,,\\
\{\mathcal{H}, \kappa_3\} &= (\Omega-\nu)\,\kappa_4\,, \qquad \{\mathcal{H}, \kappa_4\} = -(\Omega-\nu)\,\kappa_3\,,\\
\{\mathcal{H}, \kappa_5\} &= -\omega^2\,\kappa_6\,, \qquad \{\mathcal{H}, \kappa_6\} = \kappa_5\,.
\end{aligned} \tag{4.18}$$

Thus the Poisson brackets of the moment maps with each other and with the Hamiltonian $\mathcal{H}$ are the same as the Lie brackets of the generators $\mathrm{K}_{(\alpha)}$ with each other and with H, except for the central terms in (4.16).

Note that the moment maps are conserved, i.e.

$$\frac{d\kappa_{(\alpha)}}{dt} \equiv \frac{\partial \kappa_{(\alpha)}}{\partial t} + \{\kappa_{(\alpha)}, \mathcal{H}\} = 0\,. \tag{4.19}$$

In what follows, we shall focus our attention on the centre-of-mass Hamiltonian for the (x, y) plane alone. Thus we write $\mathcal{H} = \mathcal{H}_\perp + \mathcal{H}_3$, where, from (4.17), we have

$$\begin{aligned}
\mathcal{H}_\perp &= \frac{1}{2M}\Big[(\pi_x + M\nu y)^2 + (\pi_y - M\nu x)^2\Big] + \tfrac{1}{2}M\omega^2(x^2 + y^2)\,, &(4.20)\\
\mathcal{H}_3 &= \frac{1}{2M}\,\pi_z^2 + \tfrac{1}{2}M\omega^2 z^2\,. &(4.21)
\end{aligned}$$

We may write $\mathcal{H}_\perp$ as

$$\begin{aligned}
\mathcal{H}_\perp &= \frac{\Omega+\nu}{4M\Omega}(\kappa_1^2 + \kappa_2^2) + \frac{\Omega-\nu}{4M\Omega}(\kappa_3^2 + \kappa_4^2)\,,\\
&= \frac{\Omega+\nu}{4M\Omega}\Big[(\pi_x + M\Omega y)^2 + (\pi_y - M\Omega x)^2\Big]\\
&\quad + \frac{\Omega-\nu}{4M\Omega}\Big[(\pi_x - M\Omega y)^2 + (\pi_y + M\Omega x)^2\Big]\,.
\end{aligned} \tag{4.22}$$

It will be convenient to define the complex combinations

$$\begin{aligned}
a &= (\pi_x + M\Omega y) + i(\pi_y - M\Omega x)\,,\\
a^\dagger &= (\pi_x + M\Omega y) - i(\pi_y - M\Omega x)\,,\\
b &= (\pi_x - M\Omega y) - i(\pi_y + M\Omega x)\,,\\
b^\dagger &= (\pi_x - M\Omega y) + i(\pi_y + M\Omega x)\,,
\end{aligned} \tag{4.23}$$

in terms of which we may write the Hamiltonian as

$$\mathcal{H}_\perp = \frac{\Omega+\nu}{4M\Omega}\,a^\dagger a + \frac{\Omega-\nu}{4M\Omega}\,b^\dagger b\,. \tag{4.24}$$

Note that the angular momentum $J = x\pi_y - y\pi_x$ is given by

$$J = \frac{1}{4M\Omega}\,(b^\dagger b - a^\dagger a)\,. \tag{4.25}$$

5 Quantisation

We may pass from the classical algebra of Poisson brackets to the quantum commutation relations by means of the standard replacement

$$\boldsymbol{\pi} \longrightarrow \hat{\boldsymbol{\pi}} = -i\hbar\boldsymbol{\nabla}\,. \tag{5.1}$$

One then verifies that with the replacements

$$\{A, B\} \longrightarrow \widehat{\{A, B\}} = \frac{i}{\hbar}\,[\hat{A}, \hat{B}]\,, \tag{5.2}$$

the Poisson bracket algebra given in (4.16) and (4.18) yields the same commutator algebra as that of the quantum operators.

The quantities a and b defined in (4.23) now become the quantum operators

$$\begin{aligned}
\hat{a} &= (\hat{\pi}_x + M\Omega y) + i(\hat{\pi}_y - M\Omega x)\,,\\
\hat{a}^\dagger &= (\hat{\pi}_x + M\Omega y) - i(\hat{\pi}_y - M\Omega x)\,,\\
\hat{b} &= (\hat{\pi}_x - M\Omega y) - i(\hat{\pi}_y + M\Omega x)\,,\\
b^\dagger &= (\hat{\pi}_x - M\Omega y) + i(\hat{\pi}_y + M\Omega x)\,,
\end{aligned} \tag{5.3}$$

obeying the commutation relations of two mutually commuting sets of annihilation and creation operators:

$$[a, a^\dagger] = [b, b^\dagger] = 4\hbar M\Omega\,, \tag{5.4}$$

and the Hamiltonian and angular momentum can be written as

$$\hat{\mathcal{H}}_\perp = \frac{\Omega+\nu}{4M\Omega}\,\hat{a}^\dagger\hat{a} + \frac{\Omega-\nu}{4M\Omega}\,\hat{b}^\dagger\hat{b} \;+\hbar\Omega\,, \tag{5.5}$$

$$J = \frac{1}{4M\Omega}\,(\hat{b}^\dagger\hat{b} - \hat{a}^\dagger\hat{a})\,. \tag{5.6}$$

There is therefore a unique ground state $|0\rangle$ of energy $\hbar\Omega$ satisfying $a|0\rangle = 0$, $b|0\rangle = 0$, with excited states of the form

$$|p, q\rangle = (a^\dagger)^p\,(b^\dagger)^q\,|0\rangle\,, \tag{5.7}$$

and energies

$$E_{p,q} = (p+q+1)\hbar\Omega + (p-q)\hbar\nu\,. \tag{5.8}$$

The state $|p, q\rangle$ has angular momentum

$$J_{p,q} = (q-p)\hbar\,. \tag{5.9}$$

These results coincide with those of [7] and [8].

Concretely, if we write $\zeta = x + iy$, then (setting $\hbar = 1$ for convenience)

$$\begin{aligned}
\hat{a} &= -2i(\bar{\partial} + \tfrac{1}{2}M\Omega\zeta)\,, \qquad & \hat{a}^\dagger &= -2i(\partial - \tfrac{1}{2}M\Omega\bar{\zeta})\,,\\
\hat{b} &= -2i(\partial + \tfrac{1}{2}M\Omega\bar{\zeta})\,, \qquad & \hat{b}^\dagger &= -2i(\bar{\partial} - \tfrac{1}{2}M\Omega\zeta)\,,
\end{aligned} \tag{5.10}$$

where $\partial \equiv \partial/\partial\zeta$ and $\bar\partial \equiv \partial/\partial\bar\zeta$. The wave functions are of the form

$$\Psi_{p,q} \propto \bar\zeta^p\, \zeta^q\, e^{-\frac{1}{2}M\Omega\zeta\bar\zeta}\,. \tag{5.11}$$

The angular momentum operator now takes the form

$$J = \zeta\partial - \bar\zeta\bar\partial\,. \tag{5.12}$$

6 The Lightlike Lift

By a result of Eisenhart [16], a classical Hamiltonian of the form

$$\mathcal{H} = \frac{1}{2m}\, g^{IJ}\,(\pi_I - eA_I)(\pi_J - eA_J) + V(x) \tag{6.1}$$

may be obtained by reduction from the Hamiltonian

$$\widetilde{\mathcal{H}} = g^{IJ}\left(\pi_I - \frac{e}{m}\,A_I\pi_s\right)\left(\pi_J - \frac{e}{m}\,A_J\pi_s\right) + 2\pi_s\pi_t + \frac{2}{m}\,V\,\pi_s^2\,, \tag{6.2}$$

for a massless particle moving in the higher-dimensional Lorentzian metric

$$d\tilde{s}^2 = g_{IJ}dx^I dx^J + \frac{2e}{m}\,A_I dx^I dt + 2dtdv - \frac{2}{m}V\,dt^2\,. \tag{6.3}$$

The procedure is to impose the massless condition $\widetilde{\mathcal{H}} = 0$ and substitute for the constant component of momentum π_s associated to the null Killing vector $\partial/\partial v$ the value $\pi_v = m$, and to identify the moment map π_t associated to time translations with $-\mathcal{H}$.

At the quantum level, the Schrödinger equation associated with the quantised Hamiltonian $\hat H$ can be derived from the massless Klein-Gordon equation $\tilde\Box\Phi = 0$ in the higher-dimensional metric (6.3), by writing

$$\Phi(x,t,v) = e^{imv}\,\Psi(x,t)\,, \tag{6.4}$$

thus giving

$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2m}\,(\nabla_I - ieA_I)(\nabla_J - ieA_J)\Psi + V\Psi \tag{6.5}$$

in the lower dimension.

In our case $g_{IJ} = \delta_{IJ}$, $x^I = x_{ia}$ and

$$V = \tfrac{1}{2}m\omega^2 \sum_a \mathbf{r}_a^2 + \sum_{a,b} V(\mathbf{r}_a - \mathbf{r}_b)\,. \tag{6.6}$$

The metric (6.3) can be written as $d\tilde{s}^2 = ds^2 + ds^2_{\rm rel}$, where

$$\begin{aligned} ds^2 &= N\Big[(dx - \nu y dt)^2 + (dy + \nu x dt)^2 + dz^2 - [\Omega^2(x^2+y^2) + \omega^2 z^2)]dt^2\Big] \\ &\quad +2dtdv\,, \end{aligned} \tag{6.7}$$

$$\begin{aligned} ds^2_{\rm rel} &= \frac{1}{N}\sum_{a<b}\Big[(dx_a - dx_b - \nu(y_a-y_b)dt)^2 + (dy_a - dy_b + \nu(x_a - x_b)dt)^2 \\ &\quad +(dz_a - dz_b)^2 - \{\Omega^2(x_a-x_b)^2 + \Omega^2(y_a-y_b)^2 + \omega^2(z_a-z_b)^2 \\ &\qquad +\frac{2}{m}V(\mathbf{r}_a - \mathbf{r}_b)\}dt^2\Big]\,. \end{aligned} \tag{6.8}$$

Note that although we have written the metric as the sum of two terms, it is not a product metric. It could be made into a product metric at the expense of introducing another timelike coordinate τ, so that t in (6.8) is replaced by τ. If $V(\mathbf{r}_a - \mathbf{r}_b)$ is positive, then ds^2 and $ds^2_{\rm rel}$ would then both be Lorentzian.

The centre of mass metric ds^2, and indeed the entire metric $d\tilde{s}^2$, is invariant under the action of the Killing vectors $\widetilde{\rm K}_{(\alpha)}$, $\widetilde{\rm H}$ and J which are the lifts of the vector fields ${\rm K}_{(\alpha)}$, H and J that we introduced in section 2, where

$$\begin{aligned} \widetilde{\rm K}_1 &= {\rm K}_1 + N\Omega\,[x\sin(\Omega+\nu)t + y\cos(\Omega+\nu)t\,]\,\frac{\partial}{\partial v}\,, \\ \widetilde{\rm K}_2 &= {\rm K}_2 + N\Omega\,[y\sin(\Omega+\nu)t - x\cos(\Omega+\nu)t\,]\,\frac{\partial}{\partial v}\,, \\ \widetilde{\rm K}_3 &= {\rm K}_3 + N\Omega\,[x\sin(\Omega-\nu)t - y\cos(\Omega-\nu)t\,]\,\frac{\partial}{\partial v}\,, \\ \widetilde{\rm K}_4 &= {\rm K}_4 + N\Omega\,[y\sin(\Omega-\nu)t + x\cos(\Omega-\nu)t\,]\,\frac{\partial}{\partial v}\,, \\ \widetilde{\rm K}_5 &= {\rm K}_5 + N\omega z\sin\omega t\,\frac{\partial}{\partial v}\,, \\ \widetilde{\rm K}_6 &= {\rm K}_6 - Nz\cos\omega t\,\frac{\partial}{\partial v}\,, \\ \widetilde{\rm H} &= {\rm H}\,, \\ \widetilde{\rm J} &= {\rm J}\,. \end{aligned} \tag{6.9}$$

The Lie brackets of $\widetilde{\rm H}$ and $\widetilde{\rm J}$ with $\widetilde{\rm K}_{(\alpha)}$ are the same as in (2.4) and (2.7), but the $\widetilde{\rm K}_{(\alpha)}$ no longer commute with each other. Their Lie brackets

$$[\widetilde{\rm K}_1, \widetilde{\rm K}_2] = -2N\Omega\frac{\partial}{\partial v}\,, \qquad [\widetilde{\rm K}_3, \widetilde{\rm K}_4] = 2N\Omega\frac{\partial}{\partial v}\,, \qquad [\widetilde{\rm K}_5, \widetilde{\rm K}_6] = -N\frac{\partial}{\partial v} \tag{6.10}$$

coincide with the negatives of the Poisson brackets of the moment maps $\kappa_{(\alpha)}$, as given in (4.16). In fact, one may pass from the moment maps $\kappa_{(\alpha)}$ to the Killing vectors $\widetilde{\rm K}_{(\alpha)}$ by

means of the replacements

$$\{\pi_x, \pi_y, \pi_z, m\} \longleftrightarrow \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial v}\right\}. \tag{6.11}$$

In accordance with Larmor's theorem, the apparent dependence of the metrics (6.8) on the Larmor frequency ν may be eliminated and ν set to zero, by the coordinate transformation (2.8) with $\varpi = -\nu$. The centre of mass metric restricted to $z = 0$ then takes the simple form

$$ds^2 = N\Big[dx^2 + dy^2 + (\Omega^2(x^2 + y^2) + \omega^2 z^2)dt^2\Big] + 2dtdv\,. \tag{6.12}$$

We shall discuss this further in the next section.

7 Brdička-Eardley-Nappi-Witten waves

In this section we shall relate the Eisenhart lift we have found above to the general class of pp-wave solutions to the Einstein-Maxwell equations that is defined by two arbitrary holomorphic functions, $f(u, \zeta)$ and $\phi(u, \zeta)$, with complex conjugates $\bar{f}(u, \bar{\zeta})$ and $\bar{\phi}(u, \bar{\zeta})$. The configuration takes the form

$$ds^2 = du\Big(dv + H(u, \zeta, \bar{\zeta})du\Big) + d\zeta d\bar{\zeta}, \qquad F = du \wedge (\partial_\zeta \phi d\zeta + \partial_{\bar{\zeta}} \bar{\phi} d\bar{\zeta}), \tag{7.1}$$

where

$$H(u, \zeta, \bar{\zeta}) = f(u, \zeta) + \bar{f}(u, \bar{\zeta}) - 2\phi(u, \zeta)\bar{\phi}(u, \bar{\zeta}), \tag{7.2}$$

and

$$F \equiv F_{\mu\nu} dx^\mu \wedge dx^\nu\,. \tag{7.3}$$

We have used light-cone coordinates (u, v) and complex coordinates $(\zeta, \bar{\zeta})$. The covariantly-constant null vector field is $l^\mu \partial_\mu = \partial/\partial v$.

A very special property of this solution is that the Maxwell and Riemann tensors can be written as

$$F^{\mu\nu} = l^{[\mu} s^{\nu]}, \qquad R^{\mu\nu\alpha\beta} = l^{[\mu} k^{\nu][\beta} l^{\alpha]}. \tag{7.4}$$

The vector s^μ and symmetric tensor $k^{\mu\nu}$ are orthogonal to l^μ. Therefore the solution will solve virtually any effective action with higher-order curvature terms. This includes closed-string corrections of the form $(R_{\mu\nu\alpha\beta})^n$, open-string corrections of the Born-Infeld type and mixed "corrections" introducing non-minimal coupling. The solutions might however be renormalized by higher-derivative corrections. For the gravitational or closed-string corrections this has been discussed in [21].

A weaker constraint that implies vanishing of some closed-string corrections is the one of conformal flatness. The Weyl tensor vanishes for (7.1) as long as $\partial_\zeta^2 H = \partial_{\bar\zeta}^2 H = 0$ (compare with the vacuum equation $\partial_\zeta \partial_{\bar\zeta} H = 0$). This is exactly what we will find for the Nappi-Witten case below. It is then straightforward to find a set of coordinates where conformal flatness becomes explicit.

Some special members of this family of pp-waves are supersymmetric. The Killing spinor equation of ungauged $N=2, D=4$ supergravity,

$$\left(d + \frac{1}{4}\omega_{ab}\Gamma^{ab} - \frac{1}{4}F_{ab}\Gamma^{ab}\Gamma\right)\epsilon = 0\,, \tag{7.5}$$

reduces to

$$\Gamma^u \epsilon = 0, \qquad \partial_u \epsilon + 2F_{ux}\Gamma^x \epsilon = 0\,. \tag{7.6}$$

We have changed to real coordinates in the transverse space, $\zeta = x + iy$. These gamma matrices are flat, i.e. they obey a Clifford algebra given by (7.1) with $H = 0$. The first condition is the usual supersymmetry condition for non-electromagnetic pp-waves. The second can only be solved if F_{ux} is independent of the transverse space, i.e. if $F_{ux} = f(u)$. Then we find the non-trivial Killing spinor

$$\epsilon = e^{-2\int f(u)du\Gamma^x}\epsilon_0, \quad \Gamma^u \epsilon_0 = 0\,. \tag{7.7}$$

This wave preserves half of the vacuum supersymmetries. Notice that all supersymmetric solutions with $f(\zeta, u)$ linear in ζ are conformally flat.

A slightly more restrictive requirement is that the Maxwell field $F_{\mu\nu}$ be covariantly constant

$$\nabla_\rho F_{\mu\nu}\,. \tag{7.8}$$

It follows that $F_{ux} = C$, or equivalently $\partial_\zeta \phi(u,\zeta) = C$, and hence

$$\phi(u,\zeta) = C\zeta, \qquad H(u,\zeta,\bar\zeta) = f(u,\zeta) + \bar f(u,\bar\zeta) - 2C^2\zeta\bar\zeta\,, \tag{7.9}$$

where C is a constant. This special solution has been considered by several authors. Brdička [24] seems to have been the first to obtain it. Eardley [23] obtained it from requiring $F_{\mu\nu}$ to be covariantly constant. Nappi and Witten constructed this geometry as the target space of a Wess-Zumino-Witten conformal field theory with central charge $c = 4$ [17], although they did not state it is also a solution to Einstein-Maxwell theory. The Nappi-Witten form of the metric can be obtained from the solution (7.1) with functions (7.9) by considering $f(u,\zeta) = 0$. Performing the coordinate transformations

$$\zeta = (a_1 - ia_2)\, e^{iu'/2}, \quad u = \frac{u'}{\sqrt{8}C}, \quad v = 2\sqrt{8}C\left(v' + \frac{bu'}{2}\right)\,, \tag{7.10}$$

we obtain the metric form used by Nappi and Witten

$$ds^2 = 2du'dv' + bdu'^2 + da^i da^j \delta_{ij} + \epsilon_{ij} a_j da_i du', \quad F = \frac{du'}{\sqrt{2}} \wedge \left(\cos \frac{u'}{2} da_1 + \sin \frac{u'}{2} da_2 \right) . \quad (7.11)$$

In what follows we will drop the primes for convenience.

Despite appearances, the spacetime (7.11) is completely homogeneous. In fact, the metric (7.11) is a bi-invariant metric on the Cangemi-Jackiw group, which we denote by $\mathcal{CJ}$. This group is the universal covering group of a central extension of $\mathbb{E}(2)$, the three-dimensional isometry group of Euclidean two-space, by the additive group of real numbers $\mathbb{R}$:

$$\mathcal{CJ} = \widetilde{G_4}, \quad G_4/\mathbb{R} = \mathbb{E}(2) . \quad (7.12)$$

G_4 is of course the central extension of $\mathbb{E}(2)$, and the tilde stands for universal covering. We denote the generators of $\mathcal{CJ}$, $T_A = \{P_i, J, T\}$, for $i = 1, 2$, where the first three generate, respectively, translations and rotation in Euclidean two-space, and the last is the central element. An arbitrary element of the $\mathcal{CJ}$ group is represented as

$$g = \exp(a_i P_i) \exp(uJ + vT) . \quad (7.13)$$

The Lie algebra of the group has non-vanishing commutators

$$[P_i, P_j] = \epsilon_{ij} T, \qquad [J, P_i] = \epsilon_{ij} P_j , \quad (7.14)$$

where ϵ_{ij} is the Levi-Civita tensor density for Euclidean two-space. Since J generates rotations we have $0 \leq u \leq 2\pi$, for G_4. The other coordinates are unconstrained. Topologically therefore $G_4 \equiv S^1 \times \mathbb{R}^3$. But for $\mathcal{CJ} = \widetilde{G_4}$ we have $-\infty \leq u \leq \infty$, and therefore $\mathcal{CJ}$ is topologically $\mathbb{R}^4$, just as the pp-wave solution.

At this point we would like to remark that the normal subgroup generated by P_i, T is clearly the nilpotent Bianchi II Lie algebra, and so it is isomorphic to the Heisenberg algebra. This will be of interest in the dynamical analysis performed below.

We now give the isometry group. It was noted in [17] that it has dimension seven. Explicitly we compute the Killing vector fields to be:

$$Ł_J \equiv \partial_u, \quad R_T \equiv L_T \equiv \partial_v,$$

$$R_i \equiv \partial_i + \tfrac{1}{2}\epsilon_{ij} a_j \partial_v, \quad R_J \equiv \partial_u - \epsilon_{ij} a_j \partial_i, \quad (7.15)$$

$$L_i \equiv \cos u \partial_i + \sin u \epsilon_{ij} \partial_j + \tfrac{1}{2} \left(\sin u a_i - \cos u \epsilon_{ij} a_j \right) \partial_v .$$

These split into two copies of the G_4 Lie algebra. The non-trivial commutators are:

$$[L_J, L_i] = \epsilon_{ij} L_j, \quad [L_i, L_j] = \epsilon_{ij} L_T; \qquad [R_J, R_i] = -\epsilon_{ij} L_j, \quad [R_i, R_j] = -\epsilon_{ij} R_T \,. \tag{7.16}$$

The isometry group is therefore $(G_4^L \times G_4^R)/N_1$, where N_1 is the common centre generated by T. The right-invariant vector fields, R_i, generate the left action of the group on the spacetime. Considering an element $g' = \exp(a'_i P_i) \exp(u'J + v'T) \in G_4^L$ acting on $g = \exp(a_i P_i) \exp(uJ + vT)$, one obtains the transformation

$$(a_i, u, v) \to (a_i \cos u' - \epsilon_{ij} a_j \sin u' + a'_i, u + u', v + v' + \frac{1}{2} a'_k \epsilon_{ki} a_i \cos u' + \frac{1}{2} a'_i a_i \sin u') \,. \tag{7.17}$$

The right action generated by $g' \in G_4^R$ is obtained by interchanging primed and unprimed coordinates on the RHS of (7.17).

A set of right (left) invariant forms, i.e. dual to the right (left) invariant vector fields, is $\{\rho^a\}(\{\lambda^a\})$, given by:

$$\begin{aligned} &\rho^J = du, \quad \rho_i = da_i + \epsilon_{ij} a_j du, \quad \rho^T = dv - \frac{1}{2}\epsilon_{ij} a_j da_i - \frac{1}{2} a_i a_i du, \\ &\lambda^J = du, \quad \lambda^T = dv + \frac{1}{2}\epsilon_{ij} a_j da_i, \quad \lambda_i = \cos u da_i + \sin u \epsilon_{ij} da_j \,. \end{aligned} \tag{7.18}$$

The metric can then be written in the explicitly right-invariant or left-invariant form

$$ds^2 = \rho^T \rho^J + b\rho^J \rho^J + \rho_i \rho_i = 2\lambda^J \lambda^T + b\lambda^J \lambda^J + \lambda_i \lambda_i \,, \tag{7.19}$$

showing it is the bi-invariant metric on the group.

8 Conclusion

In this paper we have provided a fairly full account of the symmetries, both classical and quantum, of a non-relativistic system of charged particles each with the same charge to mass ratio, moving in a magnetic field and harmonic trapping potential and subject to mutual interactions depending only on their separations. The assumption of equal charge to mass ratios gives rise to a sort of relativity principle in which the Galilei and Bargmann groups are deformed to the Newton-Hooke group. We have described the action of this group on the non-relativistic spacetime which is the analogue of Newton-Cartan spacetime. In this way we have given a group theoretic interpretation to a well known theorem of Kohn. Interestingly, Larmor's theorem becomes exact in this situation and we used it to show that what appear to be a one-parameter family of deformations are in fact isomorphic.

We have also given a "relativistic" description in terms of the null geodesics in a spacetime admitting a null Killing vector field. In the planar case we have shown that this spacetime is a pp-wave and may be thought of as a bi-invariant metric on the Cangemi-Jackiw group.

We have not dwelt in this paper on the possible applications of our results, but we would like to remark that in addition to the quantum Hall effect and cyclotron resonances [2], and quantum dots [25, 26], one might expect them to be relevant for Bose-Einstein condensates and the Gross-Pitaevskii equation [27, 28], and possibly also for the collective model of nuclear physics (we owe this latter suggestion to David Khmelnitskii). In the light of current interest in laboratory analogue models which are able to capture some aspects of general relativity, cosmology and quantum gravity, this might be worth pursuing, as would possible implications for the AdS/CFT correspondence. At the more formal level it would be interesting to explore supersymmetric [29], conformal [30] and non-commutative [31, 32]. extensions of our results.

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